Extracting the Parametric Model of Duffing's Oscillator by Using a High Gain Observer

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Abstract. In this article we propose a simple high gain observer for extracting the unknown parameters of the Duffing's oscillator. It is shown that this system is observable and identifiable algebraically, with respect to a well-chosen output. Hence, an extended differential parameterization of the output and its time derivatives can be obtained. Based on these facts, we evaluate the obtained parameterization in a finite number of times to build a set of algebraic equations, and then, the parametric model is obtained by an inverse matrix. Although, the time derivatives output are not available, we overcome this difficulty by using an practical High-Gain Observer.

Keywords. Mechanical Oscillator, Chaos Reconstruction and High-Gain Observers.

1 Introduction

An interesting problem in chaos theory and its applications is the reconstruction of the unknown variables and parameters of a chaotic system. This problem is important because any experimental dynamical system posses only a few variables or parameters that can be measured (see [Parlitz *et al.*, 1994] and [Abarbanel,1996]). In some cases, it is necessary to estimate or reconstruct the unavailable quantities in order to completely determine the system's state and to achieve a good synchronization or to predict the system's behavior.

Roughly speaking, there are two approaches for reconstructing a chaotic attractor. The first relies on control theory; like the procedure based on system inversion (see [U. Feldmann *et al.*, 1996], [H. Huijberts *et. Al*, 2000] and [M.S. Suarez *et. al.*,2003]; and, traditional identification schemas and state observer design (see [G. Cheng, 1995], [A. S. Poznyak *et al.*, 1999],[I. Chavez M. *et al.*, 2002] and ,[I. Chavez M. *et al.*, 2002]). The second approach is based on time series from a particular chaotic system, which are used in the so-called time delay reconstruction of a phase space (see [Parlitz *et al.*, 1994], [K.T. Alligood *et al.*, 1997], [I. Makoto *et al.*, 1997], [T. Sauer *et al.*, 1991], [T. Stojanovski *et al.*, 1997] and [F. Takens, 1981]).

In this communication, we deal with the reconstruction of the Duffing's System (DMO) by means of measurements of the position state, which is considered as the system output. The on-line identification procedure is based on algebraic properties that the DMO satisfies ([R. Martínez *et al.*, 2001] and [I. Chavez M *et al.*, 2002]). Those properties allow finding a differential parameterization of the output and its time derivatives. Then, we evaluate the obtained parameterization in a discrete set of time to form a set of linear equations, where the parameters of the DMO are the variables of the obtained linear equations. And finally, we recover the unknown parameter by solving a set of linear equations. We should mention that time evaluation of the differential parameterization requires the unavailable output time derivatives. Such difficulty is overcome by using a high-gain observer (HGO). An HGO does not require an accurate model, and the error can be as small as desired (see [Dabroom and Khalil, 1999]), where the error is the difference between the original system measured signal and the signal estimated by the observer.

The rest of this work is organized as follows. Section 1 gives a brief description of the DMO. Section 2 is devoted to studying some important algebraic properties of the differential equations of the DMO. Also, in Section 2 we present an identification procedure by means of the previously introduced algebraic properties assuming that the time derivatives of the selected output are available. Finally, in the same Section, we present an HGO for computing the unavailable time derivatives of the output. Section 3 contains the results of the simulations while Section 4 is devoted to giving some conclusions. Finally, in the Appendix we provide a proof of Propositions 1 and 2.

2 Duffing's Mechanical Oscillator

Consider the traditional DMO, described by:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -p_1 v - p_3 x^3 - p_2 x + A\cos(wt); \end{aligned} \tag{1}$$

where x measures the oscillator position, A is the amplitude of the forcing function, w is the forcing frequency, p_1 is the damping coefficient, and p_2 and p_3 are fixed constants related to a non-linear stiffness function. When the parameters values are in a neighborhood of $\{p_1 = 0.4, p_2 = -1.1, p_3 = 1, A = 2.1, w = 1.8\}$, this system has a chaotic behavior (see [Nayfeh & Mook, 1979], [Alligood *et al.*, 1997]).

Recovering the Set of Parameters

We first establish the main problem to solve: We desire to recover the unknown parameters $\{p_1, p_2, p_3, A\}$ from the measured output or available position x. To solve it, we first discuss two important and useful definitions which will allow us to transform the original system into a set of differential parameterizations of the output. The differential parameterization of the output is obtained based on its successive output time derivatives; where the order of the time derivative of the output is a function of the number of parameters that we want to identify[1]. Finally, evaluating the parameterization in a finite number of times, it is possible to recover the unknown parameters by means of the inverse of a matrix.

Some Algebraic Properties

We say that a system is algebraically observable if there exists a suitable output provided that all the system variables can be differentially parameterized solely in terms of the output. Moreover, if we can express the parameter vector as a parametric function of the output and a finite number of its time derivatives, we say that the system is identifiable with respect to this output.

Now, let us consider again the **DMO** and define the output $y = x$. Evidently, we have:

$$\begin{aligned} v &= \dot{y}, \\ \dot{v} &= -vp_1 - yp_2 - y^3 p_3 + A\cos(wt). \end{aligned} \tag{2}$$

If we continue to differentiate the output with respect to time, we obtain:

$$\begin{aligned} \ddot{y} &= -\dot{y}p_1 - yp_2 - y^3 p_3 + A\cos(wt), \\ y^{(3)} &= -\ddot{y}p_1 - \dot{y}p_2 - 3y^2\dot{y}p_3 - wA\sin(wt), \\ y^{(4)} &= -y^{(3)}p_1 - \ddot{y}p_2 + \left(-3y^2\ddot{y} - 6y\dot{y}^2\right)p_3 - w^2 A\cos(wt), \\ y^{(5)} &= -y^{(4)}p_1 - y^{(3)}p_2 - (-18y\dot{y}\ddot{y} - 3y^2 y^{(3)} - 6\dot{y}^3)p_3 + w^3 A\sin(wt). \end{aligned}$$

After some manipulations, the last set of equations may be rewritten as:

$$M(y)Q = F(w,t) \tag{3}$$

Where Q stands for the vector of parameters defined as:

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[1] In this case is necessary to obtain from $y^{(1)}$ to $y^{(5)}$ because we want to estimate four unknown parameters.

$$Q^T \triangleq [q_1, q_2, q_3, q_4] = \left[\frac{1}{A}, \frac{p_1}{A}, \frac{p_2}{A}, \frac{p_3}{A}\right] \tag{4}$$

$M(y)$ is the matrix of the output y and its time derivatives, defined by:

$$M(y) \doteq \begin{bmatrix} -\ddot{y} & -\dot{y} & -y & -y^3 \\ -y^{(3)} & -\ddot{y} & -\dot{y} & -3y^2\dot{y} \\ -y^{(4)} & -y^{(3)} & -\ddot{y} & (-3y^2\ddot{y} - 6y\dot{y}^2) \\ -y^{(5)} & -y^{(4)} & -y^{(3)} & (-18y\dot{y}\ddot{y} - 3y^2y^{(3)} - 6\dot{y}^3) \end{bmatrix}. \tag{5}$$

$F(w,t)$ is the independent vector of the variable y whose components are given as:

$$F(w,t) \triangleq [-\cos(wt), w\sin(wt), w^2\cos(wt), -w^3\sin(wt)] \tag{6}$$

Therefore, vector Q is algebraically identifiable with respect to the variable $y(t)$. The relation (3) is referred as the extended differential parameterization.

Next, let us consider the following assumptions:

A1: *The solution $y(t)$ and its time derivatives exhibit a chaotic behavior for the system (1).*

A2: *The time derivatives of the selected output are always available.*

Now, we mention an important proposition that allows us to compute the unknown parameter vector.

Proposition 1: *Let us consider system (1) with its respective extended differential parameterization (3), under assumptions A1 and A2. Then, the inverse of matrix (5) exists almost for any time.*

Proof: (Refer to Appendix).

Notice that, substituting the vector values Q into relation (4), we can recover the set of unknown parameters $\{p_1, p_2, p_3, A\}$. Finally, in next section, we propose a practical numerical differentiator to estimate the time derivatives of the variable y.

A simple HGO

In order to obtain the time derivatives of the output y, we suggest the following scheme to estimate the time derivatives. Let us define vector $Y^T=[y \ldots y^{(5)}]$ and let us propose the following filter given by:

$$\dot{\hat{Y}} = A\hat{Y} + HC(Y - \hat{Y}) \tag{7}$$

Where, matrix A defined the well-known Brunovsky form [Dabroom & Khalil, 1999] and

$$H^T = \left[\frac{\alpha_1}{\varepsilon}, \ldots, \frac{\alpha_6}{\varepsilon^6}\right], C = [1,0,\ldots,0] \tag{8}$$

ε is a small positive parameter and the positive constants α_i are selected such that the polynomial defined as:

$$p(s) = s^6 + \alpha_1 s^5 + \ldots + \alpha_6, \tag{9}$$

is Hurwitz (see [Dabroom & Khalil, 1999] for more details).

The following proposition allows us to compute the error $\xi = Y - \hat{Y}$.

Proposition 2: *Consider the system (7) under assumptions A1. Then, the HGO proposed in (8) is able to recover Y with bounded error*

$$\|\xi\| \leq \beta n \varepsilon / \lambda^* \tag{10}$$

Where λ^* is given by

$$\lambda^* = \min\{\mathrm{Re}[roots(p(s))]\}, \tag{11}$$

β *is a positive constant which depends on the initial conditions* $\xi(0)$, *and* $n = \max_{t \in [0,\tau]} |y^{(5)}(t)|$

Notice that, we substitute the estimated time derivative $\hat{y}^{(k)}$ instead of $y^{(k)}, k = \{1,\ldots,5\}$ into expression (5).

Numerical Simulations

We first test the efficiency of the HGO by computer simulations. The experiments were implemented by using the 4th-order Runge-Kutta algorithm. The computation was performed with a precision of *8* decimal digit numbers, from $t = 0$ seconds to $t = 10$ seconds. To obtain a good performance, the step size in the numerical method was set to *0:0001*. The **DMO** parameter values were set as $p_1 = 0.3$; $p_2 = -1.2$; $p_3 = 1$, $A = 1.8$, $w = 1.9$. The initial conditions were set as $y(0) = 1$ and $\dot{y}(0) = -1$. The polynomial was chosen to be $p(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)$, with $\zeta = 0.707$ and $\omega_n = 0.9$. The gain of the HGO was selected as $\varepsilon = 0.005$.

For this particular simulation after $t = 0.25$ seconds the derivative estimation errors were around:

$$\xi_1 = 0.1x10^{-6},\ \xi_2 = 0.3x10^{-5},\ \xi_3 = 0.9x10^{-5},\ \xi_4 = 0.25x10^{-4},\ \xi_5 = 0.1x10^{-3}.$$

Consequently, we obtain a very good estimation of non available derivatives.

Finally, we probe the effectiveness of the described identification method by numerical simulations. The initial conditions and the physical parameters were taken as in the previous experiment.

Figures 1 and 2 show the estimation of the parameter A, p_1 and p_2 and p_3, respectively.

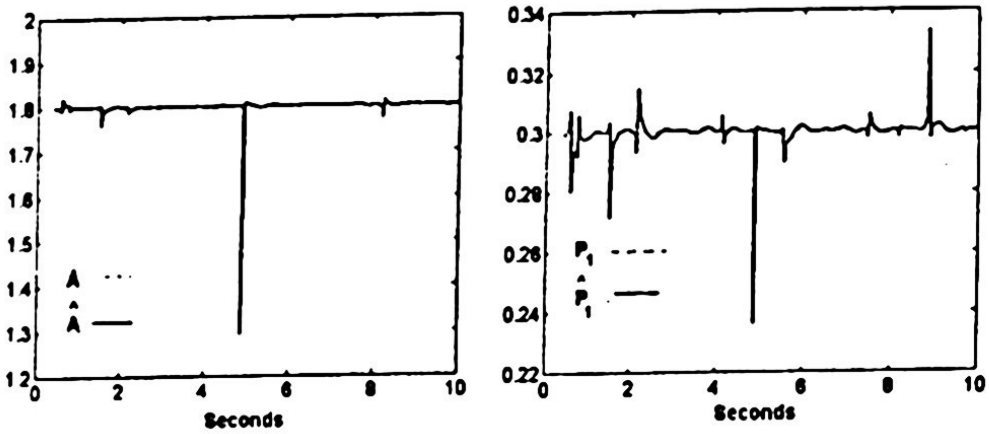

Fig. 1. Identification of parameters A and p_1

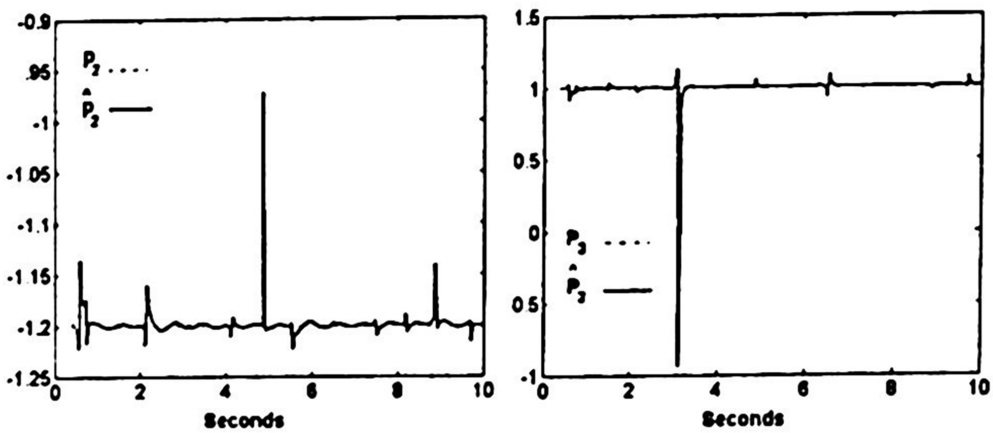

Fig. 2. Identification of parameters p_2 and p_3

3 Conclusions

Based on the algebraic differential approach for the identification problem of the traditionally Duffing's oscillator that has been treated in this paper. The fact that the system is algebraically observable and algebraically identifiable, with respect to a specified variable, allowed us to describe the original system by means of a differential parameterization of the output and its respective time derivatives. This differential parameterization has all the necessary information to recover the parametric model of the DMO. Then, we proceeded to evaluate the differential parameterization in a finite number of times in order to form a set of algebraic equations, where the unknown parameters of the DMO were obtained by computing an inverse matrix. The lack of the output's time derivatives required in the set of algebraic equations was overcome by the design of an HGO, where the observation errors can be as small as needed by tuning a specific parameter accordingly to Proposition 2.

The obtained identification solution was illustrated in a numerical simulation, where the HGO recovers the output's time derivatives, and then, the corresponding unknown parameters can be revealed.

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Appendix

Proof of Proposition 1:

We first suppose that the set of functions $\{y(t), \dot{y}(t), y(t), y^{(3)}(t)\}$ is linearly dependent in a time interval I where $y(t)$ shows a chaotic behavior according to A1. This implies the existence of a set of constants c_1, c_2, c_3, and c_4, different from zero such that:

$$c_1 y(t) + c_2 \dot{y}(t) + c_3 y(t) + c_4 y^{(3)}(t) = 0$$

Notice that if $c_1 = 0$ then the last second order differential equation turns into a first order differential equation, therefore, $y(t)$ is a monotonic decreasing or monotonic increasing function (see p. 332 of [Alligood *et al.*, 1997]) This case is not possible, because $y(t)$ has a chaotic behavior. Also, by the Poincaré-Bendixon theorem (see p. 337 of [Alligood *et al.*, 1997]) it is well-known that a second order differential equation that does not depend on time cannot exhibit a chaotic behavior. Thus, $\{y(t), \dot{y}(t), y(t), y^{(3)}(t)\}$ is linearly independent in a time interval I.

Proof of Proposition 2:

Evidently, vector $\dot{Y}$ can be written as:

$$\dot{Y} = AY + \delta_y . \tag{12}$$

With $\delta^T = [0, 0, \ldots, y^{(6)}]$ Subtracting (12) from (7), we obtain the following differential equation of the error:

$$\dot{\xi} = [A - HC]\xi + \delta_y. \tag{13}$$

Notice that the characteristic polynomial of $A = A - HC$ is given by $p(s\,\varepsilon)$, which is also Hurtwitz. That is, the proposed H assigns the eigenvalues of A at $1/\varepsilon$ times the roots of $p(s)$ (9). Hence, the error ε satisfies

$$\varepsilon(t) = e^{A(t-t_0)}\left(\varepsilon(0) + \int_{t_0}^{t} e^{A(t_0-s)}\delta_y(s)ds\right). \tag{14}$$

Since A is exponentially stable and the signal $y^{(3)}$ is bounded, we also have the following inequality:

$$\|\xi\| \le \beta e^{-\lambda^* t/\varepsilon}\|\xi\| + \beta\eta\varepsilon\left(1 - e^{-\lambda^* t/\varepsilon}\right)\lambda^* \to \beta\eta\varepsilon/\lambda^*. \tag{15}$$

Where the positives constants β, λ^*, η are previously defined in **Proposition 2.**

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